MATH 551 - Problem Set 4

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7.2 We have that A' is the midpoint of BC, B' is the midpoint of CA, and C' is the midpoint of AB. So this means that we can write $A' = (\frac{1}{2})B + (\frac{1}{2})C$, $B' = (\frac{1}{2})C + (\frac{1}{2})A$, $C' = (\frac{1}{2})A + (\frac{1}{2})B$. We will use these equalities later. First, let's unpack the equation given (by distributing the dot product). We achieve

$$(D\cdot C - D\cdot B - A'\cdot C + A'\cdot B) + (D\cdot A - D\cdot C - B'\cdot A + B'\cdot C) + (D\cdot B - D\cdot A - C'\cdot B + C'\cdot A)$$

We see that some terms cancel so we have

$$(-A'\cdot C + A'\cdot B) + (-B'\cdot A + B'\cdot C) + (-C'\cdot B + C'\cdot A)$$

We may then plug in our values for A', B', and C' to get $(-(\frac{1}{2}B + \frac{1}{2}C)\cdot C + (\frac{1}{2}B + \frac{1}{2}C)\cdot B) + (-(\frac{1}{2}C + \frac{1}{2}A)\cdot A + (\frac{1}{2}C + \frac{1}{2}A)\cdot C) + (-(\frac{1}{2}A + \frac{1}{2}B)\cdot B + (\frac{1}{2}A + \frac{1}{2}B)\cdot A)$ but we can see that these terms cancel out to 0, so we have shown that the given dot product is equal to 0. □

7.3 Let's call the midpoint of AB 'J', of AD 'M', of BC 'K', and of CD 'L'. So J connects with L and M connects with K. Let's start with the converse, that is, given AC perpendicular to BD, we must show that $|MK| = |JL|$. Well, because M, K, J, L are midpoints, we know that we can write them in terms of the lines they lie on, so we have $J = \frac{1}{2}B + \frac{1}{2}A$, $M = \frac{1}{2}A + \frac{1}{2}D$, $L = \frac{1}{2}D + \frac{1}{2}C$, and $K = \frac{1}{2}C + \frac{1}{2}B$. We may write

$$JL = L - J = (\frac{1}{2}D + \frac{1}{2}C) - (\frac{1}{2}B + \frac{1}{2}A) = \frac{1}{2}(D - B) + \frac{1}{2}(C - A)$$

and similarly, for MK we have

$$MK = K - M = (\frac{1}{2}C + \frac{1}{2}B) - (\frac{1}{2}A + \frac{1}{2}D) = \frac{1}{2}(C - A) + \frac{1}{2}(B - D)$$

Now, notice that both JL and MK share a $\frac{1}{2}(C - A)$ term, and their other term differs by a sign. Because we're only concerned about the magnitude of JL and MK, these sign differences don't matter. So, we can conclude that their magnitudes are equal. □

Now, let us consider the original direction. We must show that if $|MK| = |JL|$, then we must have AC perpendicular to BD. We may write

$$JL = L - J = (\frac{1}{2}D + \frac{1}{2}C) - (\frac{1}{2}B + \frac{1}{2}A) = \frac{1}{2}(D - A) + \frac{1}{2}(C - B)$$

similarly, for MK we have

$$MK = K - M = (\frac{1}{2}C + \frac{1}{2}B) - (\frac{1}{2}A + \frac{1}{2}D) = \frac{1}{2}(C - D) + \frac{1}{2}(B - A)$$

. But we know that the difference of the magnitudes of JL and MK is 0, so (because $C - D = DC, B - A = AB$, etc) we have that

$$0 = mag((\frac{1}{2}DC + \frac{1}{2}AB) - (\frac{1}{2}AD + \frac{1}{2}BC))$$

but we notice that this may be rearranged to be $0 = \frac{1}{2}(DC + AB - AD - BC)$. Now, we know that the dot product $AC \cdot BD = (C - A) \cdot (D - B) = CD - CB - AD + AB$. which is exactly our previous result $(\frac{1}{2}(DC + AB - AD - BC))$ except with a variation of sign. But, because we are only concerned with the magnitude, the signs do not matter. So therefore we have shown that the dot product $AC \cdot BD = 0$, which means that AC and BD are parallel. □

8.1 (See diagram below on next page.) We begin by drawing a line between B and D, making the triangle $\triangle ABD$. We now (first) consider the triangle above this BD line. We call the midpoint of BD the point M, and connect M with S and P. Now, our first subproblem is to prove that MS and MP are of equal length, and are perpendicular. We draw vectors u and v, and notice that MS can be written as $\frac{u+v}{2}$. Then we may apply a 90 degree rotation to the right (written as R), which can be written as $R(MS) = R(\frac{u+v}{2}) = \frac{R(u)+R(v)}{2} = \frac{CA+BZ}{2}$. But, we know that the average of CA and BZ (which are both sides of the P square), so we know that their average will be the line to the midpoint of the square, MP. Now, we've shown that MS and MP are the same length. We also know that MS and MP are perpendicular, because we received MP when we rotated MS by 90 degrees. Now by similarity, the same can be shown for the bottom half (M connecting to R and Q instead of S and P). So we receive that RM and QM are equal and perpendicular as well. Now we need to show that the line segments SM and MQ are part of the same line (and similarly that PM and MR are part of the same line). We do so by considering the side case MP and MQ (and by similarity these work out to be 90 degree rotations of each other). So we combining these results, we know that SQ and PR are perpendicular, and because we've shown that $SM = PM$ and $MQ = MR$, we know that their sum is equal, that is $SQ = PR$, and we are done. □

Diagram.